Facit til kursusgang 20: Vektorer i rummet 1

1. Svarene er:

$$\begin{bmatrix}21\\3\\12\end{bmatrix},\qquad \begin{bmatrix}-2\\1\\-8\end{bmatrix},\qquad \begin{bmatrix}9\\0\\12\end{bmatrix},\qquad \begin{bmatrix}5\\2\\-4\end{bmatrix},\qquad \sqrt{66},\qquad \sqrt{69},\qquad 45,\qquad \begin{bmatrix}12\\-48\\-9\end{bmatrix}$$

2. Nej

3. Løs ligningen:

$$\begin{bmatrix}-3\\-5\\2\end{bmatrix}\cdot\begin{bmatrix}2\\-4\\t\end{bmatrix}=0$$

   Denne har løsningen $t=-7$.

4. Prikker vi vektorerne $\vec{u}+t\vec{v}$ og $\vec{u}-t\vec{v}$ får vi ligningen

$$0=4-t^2+4-4t^2+16-t^2=24-6t^2.$$

   Denne har løsningerne $t=\pm 2$.

5. Vi har at

$$\vec{u}\times\vec{v}=\begin{bmatrix}-8\\5\\6\end{bmatrix},\quad \text{og}\quad \|\vec{u}\times\vec{v}\|=5\sqrt{5}.$$

6. Lad

$$\vec{u}=\begin{bmatrix}u_1\\u_2\\u_3.\end{bmatrix}.$$

   Så er

$$\vec{u}\times\vec{u}=\begin{bmatrix}u_2u_3-u_2u_3\\-(u_1u_3-u_1u_3)\\u_1u_2-u_1u_2\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}.$$

7. Vi har at

$$\vec{u}\times t\cdot\vec{v}=t\begin{bmatrix}-3\\-2\\3\end{bmatrix},\quad \text{og}\quad \|\vec{u}\times t\vec{v}\|=|t|\sqrt{22}.$$

   Dermed har parallelogrammet areal 3 for $t=\pm\frac{3}{\sqrt{22}}$.

8. Svarene er

$$\begin{bmatrix}-2\\-6\\-4\end{bmatrix},\qquad 3\begin{bmatrix}1\\-3\\-1\end{bmatrix},\qquad 9,\qquad \begin{bmatrix}2\\6\\4\end{bmatrix},\qquad 3\begin{bmatrix}1\\-3\\-1\end{bmatrix},\qquad -18.$$

9. Vi har at

$$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + u_3 v_3 = v_1 u_1 + v_2 u_2 + v_3 u_3 = \vec{v} \cdot \vec{u}.$$

10. Vi har at

$$\|k\vec{u}\| = \sqrt{k^2 u_1^2 + k^2 u_2^2 + k^2 u_3^2} = \sqrt{k^2}\sqrt{u_1^2 + u_2^2 + u_3^2} = |k|\|\vec{u}\|.$$

11. Bemærk først at $\vec{u} \cdot \vec{u} = u_1^2 + u_2^2 + u_3^2 = \|\vec{u}\|^2$. Bruger vi dette får vi

$$\begin{aligned}
\|\vec{u} + \vec{v}\|^2 &= (\vec{u} + \vec{v}) \cdot (\vec{u} + \vec{v}) = (u_1 + v_1)^2 + (u_2 + v_2)^2 + (u_3 + v_3)^2 \\
&= u_1^2 + v_1^2 + 2u_1 v_1 + u_2^2 + v_2^2 + 2u_2 v_2 + u_3^2 + v_3^2 + 2u_3 v_3 \\
&= \|\vec{u}\|^2 + \|\vec{v}\|^2 + 2(\vec{u} \cdot \vec{v}).
\end{aligned}$$

12. Lad

$$\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \quad \text{og} \quad \vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}.$$

Så er

$$\vec{u} \times \vec{v} = \begin{bmatrix} u_2 v_3 - u_3 v_2 \\ u_3 v_1 - u_1 v_3 \\ u_1 v_2 - u_2 v_1 \end{bmatrix}$$

og vi får

$$\vec{u} \cdot (\vec{u} \times \vec{v}) = u_1(u_2 v_3 - u_3 v_2) + u_2(u_3 v_1 - u_1 v_3) + u_3(u_1 v_2 - u_2 v_1) = 0,$$

samt

$$\vec{v} \cdot (\vec{u} \times \vec{v}) = v_1(u_2 v_3 - u_3 v_2) + v_2(u_3 v_1 - u_1 v_3) + v_3(u_1 v_2 - u_2 v_1) = 0.$$